MMP Learning Seminar

Week 79.

Introduction to

Anti-pluricanonical systems on Fano varieties.

X smooth projective of dimension d.

Canonically polarized: K_X is ample. (Hacon - McKernan, 2006).

$|mK_X|$ defines a birational map for $m \geq m(d)$

Calabi-Yau: $K_X \equiv 0$ then $K_X \sim_{\mathbb{Q}} 0$.

$mK_X \sim 0$ for some m that depends on X

conjecturally, m should only depend on d (open).

Fano: $-K_X$ ample. $|-mK_X|$ defines a bir map for $m \geq m(d)$

Can we find an element of $|-mK_X|$ with nice singularities?

Can we control m only using d. (Birkar, 2016).

smooth

$\Downarrow$

terminal

$\Downarrow$

1-lc

Theorem 1.1 (Effective non-vanishing): d a natural number.

There exists $m := m(d)$. such that if X is a Fano variety of dim d with klt singularities, then the linear system $|-mK_X| \neq \emptyset$.

Furthermore, the linear system contains $M \in |-mK_X|$ for which $(X, \frac{1}{m}M)$ has lc singularities.

all log discrepancies are $\geq \varepsilon$.

Theorem 1.2 (Effective birationality for ε-lc Fano varieties):

Let d be a natural number and $\varepsilon > 0$. There $m := m(d, \varepsilon)$. such that if X is ε-lc weak Fano variety of dimension d, then $|-mK_X|$ defines a birational map.

this condition is necessary.

Remark: The outcome of theorem implies that $\mathrm{vol}(-K_X) \geq \frac{1}{m^d}$.

Without the ε-lc condition there are examples for which $\mathrm{vol}(-K_X)$ can be arbitrarily small or arbitrarily large

X Fano variety.

regularity of X.

$$\mathrm{reg}(X) = \max\{\dim \mathcal{D}(X,B) \mid (X,B) \text{ log CY}\} \qquad \dim \emptyset = -1.$$

$$\mathrm{coreg}(X) = \dim X - \mathrm{reg}(X) - 1$$

coregularity of X.

$$\mathrm{coreg}(\mathbb{P}^n) = 0 \qquad \mathrm{reg}(\mathbb{P}^n) = \dim \mathcal{D}(\mathbb{P}^n, H_1 + \dots + H_{n+1}) = n-1$$

$$\mathrm{coreg}(X) \in \{0, 1, 2, \dots, \dim X\}$$

closer to toric varieties.

contains all toric varieties.

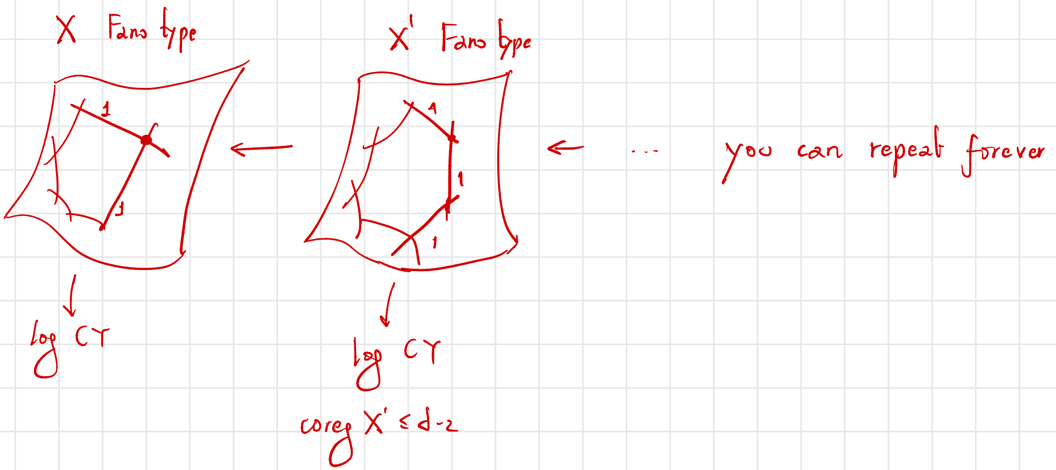
we can produce lcc with LCY structure

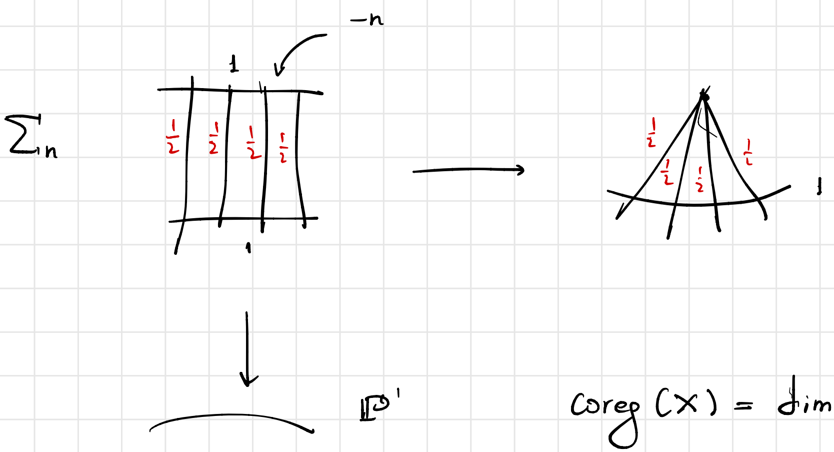
we cannot produce log canonical centers using LCY structures.

Exceptional Fano varieties $\dim X = \mathrm{coreg}(X)$

Theorem 1.3. (Boundedness of exceptional Fanos): d a natural number. The class of exceptional Fano varieties of dimension d forms a bounded family.

Remark: Fano type varieties of dimension d and coregularity $\leq d-2$ do not form a bounded family.





$\mathrm{coreg}(X) = \dim X - 1$

can be transformed to some $\mathbb{P}^1$-bundle over an exceptional Fano.

This suggest a boundedness thm for reg $= 0$ once we control certain normal bundles.

Conjecture 1.5 (BAB): Let d be a natural number & $\varepsilon > 0$. Then the set of ε-lc Fano varieties of dimension d forms a bounded family.

proved this in the toric case

Borisov – Alexeev – Borisov

proved this conjecture in dim 2.

Theorem 1.4 (Weak BAB): d is a natural number, $\varepsilon, \delta > 0$.

Consider projective varieties equipped with a boundary B s.t.:

- (X, B) is ε-lc & of dimension d.
- B is big & $K_X + B \sim_{\mathbb{R}} 0$, and
- the coefficients of B are $\geq \delta$.

Then, the set of such X forms a bounded family.

Theorem 1.6. Assume BAB holds in dimension $d-1$.

Then there is $v := v(d, \varepsilon)$ such that if X is an ε-lc weak Fano variety of dimension d, then $\mathrm{vol}(-K_X) \leq v$.

In particular, such X are birationally bounded.

Theory of complements:

choice of a LCY structure

A $\mathbb{Q}$-complement of a variety X is a boundary B with $\mathbb{Q}$-coeff for which (X,B) is lc & $K_X + B \sim_{\mathbb{Q}} 0$.

A n-complement of a variety X is a boundary B with $\mathbb{Q}$-coeff for which (X,B) is & $n(K_X+B) \sim 0$

choice of a LCY structure of index dividing n.

- Boundedness of complements
- Existence of bounded complements

Conjecture: Let d be a natural number. There exists $n := n(d)$ satisfying the following.

If X is d-dimensional & $\mathbb{Q}$-complemented, then X is N-complemented.

(open).

- Any variety with a LCY structure admits an <u>effective</u> LCY structure

Theorem 1.7 (Boundedness of complements for Fano varieties):
Let d be a natural number and $\mathcal{R} \subseteq [0,1]$ be a finite set of rational numbers. Then, there exists n, only depending on d and $\mathcal{R}$, satisfying the following:
Assume (X, B) is a projective pair such that:

- (X, B) is log canonical of dimension d,
- $B \in \Phi(\mathcal{R})$, that is, $\operatorname{coeff}(B) \in \Phi(\mathcal{R}) = \{1 - \frac{r}{m} \mid r \in \mathcal{R}, m \in \mathbb{Z}_{\geq 1}\} \cup \{0\}$
- X is of Fano type, and
- (X, B) is $\mathbb{Q}$-complemented.

Then, there exists an n-complement $K_X + B^+$ of $K_X + B$ such that $B^+ \geq B$.

Anti-pluricanonical systems on Fano varieties:

- The minimal model program.
- creating families of non-klt centers using volume
- Theory of complements
- Theory of generalized pairs. $(X, B+M)$

B: an effective divisor.

M: push-forward of a nef divisor

Sketch of the proof of 1.7 :

(X,B) with $-(K_X+B)$ $\mathbb{Q}$- complemented. of $\boxed{\text{dimension } d}$

$\operatorname{coreg}(X,B) = \dim X$ meaning all $\mathbb{Q}$-complements are klt.

In this case the singularities of (X,B) are $\boxed{\varepsilon\text{-lc}}$.

very close to one.

$$\begin{array}{c} (Y, B_Y + (1-\varepsilon)E) \\ \downarrow \\ (X,B) \end{array}$$

By ACC for lct.

(Y, B_Y+E) is again Fano type + $\mathbb{Q}$-comp.

Use ~~effective~~ birationality $\Gamma \in |-m(K_X+B)|$ the triple

$(X, B+\Gamma/m)$ is klt. //

$\underbrace{(X, B}+\Gamma)$ $\mathbb{Q}$-complement which ~~strictly~~ lc.

$\uparrow$

divisors with coeft one

$(\underbrace{Y, B_Y + \overline{E}} + \Gamma_Y)$ Y again FT. $\Longrightarrow$ MDS

Run $-(K_Y + B_Y + E)$ - MMP. reduces to:

i) $-(K_Y + B_Y + \underline{\underline{E}})$ big + nef. some component of E is FT. + KV we can lift the complement from E

ii) $K_Y + B_Y + E \equiv 0$ along a fibration $Y \longrightarrow T$.

prove the existence of a CBF for CY pairs on Fano type morphism

$\square$